Homework 1

Exercise 1. (a) Let $\Omega := [a,b] \times [c,d]$ be a rectangle. Let $F : \Omega \to \mathbb{R}^2$ be a C^1 vector field. Check the following equality

$$\int_\Omega \mathrm{curl} F dx = \int_{\partial\Omega} F \times \vec{\nu} dS.$$

Here $\vec{\nu}$ denotes the usual outward unit vector field defined in each $x \in \partial\Omega$. Recall that the curl of a vector field F is defined as $\mathrm{curl}F = \nabla \times F$. In 2 dimensions this reads as $\mathrm{curl}F = \partial_1 F^2 - \partial_2 F^1$.

(b) Prove that the previous identity holds for any set $\Omega \subset \mathbb{R}^3$ which is open, bounded, with C^1 boundary. (*Hint* use the divergence theorem from the lecture)

Exercise 2 (Integration by parts and Green's formulas)**.** In the next exercise consider Ω an open, bounded subset of $\mathbb{R}^3$ with C^1 boundary.

(a) If $u, v \in C^1(\Omega, \mathbb{R})$ prove that

$$\int_\Omega u_{x_i} v dx = \int_{\partial\Omega} uv\nu^i dS - \int_\Omega uv_{x_i} dx$$

(b) If $u, v \in C^2(\Omega, \mathbb{R})$ prove the following *Green's formulas:*

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS$$
$$\int_\Omega \nabla u \cdot \nabla v dx = -\int_\Omega u\Delta v dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} u dS$$
$$\int_\Omega u\Delta v - v\Delta u dx = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} dS$$

where $\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$ represents the derivative of u in normal direction, usually called *the normal derivative*.

Due: the 10th of October 2018